LINEAR OR NONLINEAR? STUDENTS' GRAPH REASONING AND SELECTION ON AN ONLINE ASSESSMENT

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We conducted a collective case study investigating two college algebra students' graph reasoning and selection on an online assessment. Students completed the assessment during individual, semi-structured interviews, as part of a broader validation study. The assessment contained six items; students selected Cartesian graphs to represent relationships between attributes in dynamic situations and explained their reasoning. Each student spontaneously wondered whether to select a piecewise-linear or nonlinear graph. Our qualitative analysis revealed that students' expectations about whether a graph "should" be linear or nonlinear impacted their graph selection. These expectations also influenced how they narrowed down between two graph choices that both represented the same gross covariation in attributes. We conclude with implications for course textbooks to promote students' covariational reasoning.

Keywords: Assessment, Cognition, Reasoning and Proof, Undergraduate Education

Introduction

When students sketch a Cartesian graph to represent a relationship between attributes in a dynamic situation, how do they decide if the graph should be linear or nonlinear? What role do their expectations play in this decision? For instance, a student may expect a graph to be curved because they are currently working on a unit on quadratic functions in their college algebra course. Alternatively, they may expect a graph to be linear because physical features of an object in the situation contain straight edges. We investigate how students' graph reasoning relates to their selection of a graph to represent attributes in dynamic situations, in particular when there are linear and nonlinear options.

For decades, researchers have been examining and theorizing secondary and university students' graph reasoning in dynamic situations (Carlson et al., 2002; Kerslake, 1977; Leinhardt et al., 1990). By "dynamic situation", we mean a situation involving change and variation. For example, a dynamic situation could include a cart moving on a turning Ferris wheel. We interpret "graph reasoning" broadly, to encompass students' reasoning about attributes of objects and their relationships, as well as their reasoning about observable aspects of graphs. For our purposes, we focus on Cartesian graphs. For instance, a student could expect that a graph representing the Ferris wheel situation needs to be shaped like the wheel itself. Students' graph reasoning can take on different forms depending on how students interpret attributes of objects represented in dynamic situations (Clement, 1989; Johnson et al., 2020; Moore & Thompson, 2015).

Covariational reasoning (Carlson et al., 2002) can play a role in students' interpretation of graphs as representing relationships between attributes in dynamic situations. Furthermore, covariational reasoning can engender students' differentiation between relationships represented by linear or nonlinear graphs (Paoletti & Vishnubhotla, 2022). For example, a student may choose a nonlinear graph based on their conception of relationships between the attributes of the Ferris wheel car's height and distance. Hence, students' covariational reasoning may impact their selection of a linear or nonlinear graph to represent attributes in a dynamic situation.

The two students in our study are undergraduate students enrolled in college algebra, a course known to be overly congested with content to cover in a short period of time (Gordon, 2008). Because of this, the course tends to focus on procedural applications rather than concept development, and college algebra textbooks reflect that (Mesa et al., 2012). Furthermore, college algebra has higher proportions of underrepresented minority, low income, and first-generation college students (Chen, 2016). Our study, investigating students' graph reasoning and selection, contributes to knowledge about an important, and understudied, population.

In this paper, we present findings from a collective case study (Stake, 2005) of two college algebra students who spontaneously wondered whether graphs should be piecewise-linear or nonlinear during their work on an online assessment. Our research question is: How does students' graph reasoning relate to their graph selection on a fully online assessment?

Theoretical Framework

Thompson's Theory of Quantitative Reasoning

We employ Thompson's theory of quantitative reasoning (1994, 2011) as a lens to frame this study. From Thompson's perspective, students conceive of quantities when they can think of an attribute of an object as being possible to measure. For example, consider a dynamic situation of a cart on a Ferris wheel moving counter-clockwise (Figure 1). A student can conceptualize the height of the cart as a quantity by thinking of it as being possible to measure, regardless of whether they actually measure the height of the cart or assign numerical values to measurements.

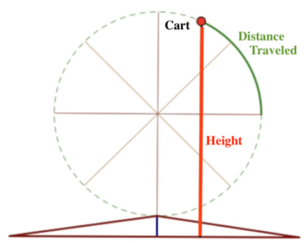

Figure 1: Ferris Wheel Dynamic Situation

When students conceive of two quantities and conceive of a relationship between them in which both quantities are capable of varying simultaneously, they are engaging in covariational reasoning (Thompson & Carlson, 2017). For example, given the Ferris wheel dynamic situation in Figure 1, a student may conceive of both height and distance traveled as quantities. They may engage in covariational reasoning by conceiving of a relationship between these two attributes in which the height fluctuates while the distance traveled increases. Thompson & Carlson put forth "gross coordination of values" as an early level of covariational reasoning. This refers to a loose connection between the direction of change in attributes. For example, a student demonstrating evidence of reasoning at this level may look at the Ferris wheel dynamic situation and say "when the height increases, the diameter decreases." This level of covariational reasoning does not necessitate identifying specific amounts of change or the rate at which the change happens.

A Graph Reasoning Framework

Johnson et al. (2020) conducted interviews with secondary students to investigate their conceptions of what graphs represent and how these conceptions change across digital task sequences. They developed a graph reasoning framework to characterize students' conceptions of what graphs represent. They included four broad forms of reasoning: Covariation, Variation, Motion, and Iconic. The first two types of reasoning, Covariation and Variation, referred to reasoning compatible with at least Thompson & Carlson's (2017) gross coordination of values and gross variation, respectively. Motion reasoning refers to students' reasoning about physical motion of objects in dynamic situations (Kerslake, 1977). Iconic reasoning referred to students' reasoning about physical features of objects in dynamic situations (Leinhardt et al., 1990). This framework encompassed students' reasoning about quantities and their relationships (Covariation, Variation), as well as their reasoning about observable aspects of dynamic situations (Motion, Iconic).

To illustrate, consider a dynamic situation involving a turning Ferris wheel, in which students select a graph to represent a relationship between two attributes in the situation: the Ferris wheel cart's height from the ground and the total distance traveled around one revolution of the wheel. Figure 2 contains a still image of the situation, along with four graph choices. Each graph is unconventional (Moore et al., 2014), with the height from the ground represented on the horizontal axis and distance traveled represented on the vertical axis. Below are examples of the four forms of graph reasoning from Johnson and colleagues' (2020) framework.

- Covariation: The Ferris wheel' height increases and decreases while the distance traveled continues to increase.
- Variation: The Ferris wheel's height increases and decreases.
- Motion: The Ferris wheel moves at a constant speed, so the graph should be straight.
- Iconic: The Ferris wheel is circular, so the graph should be curved.

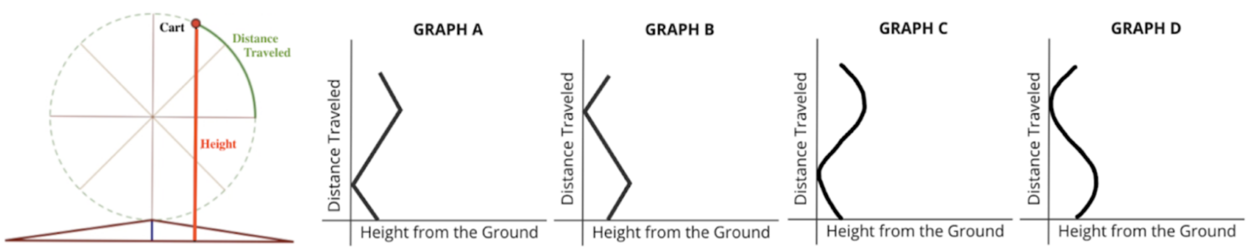

Figure 2: Ferris Wheel Dynamic Situation with Graph Choices

Students' Expectations for Graphs

When interpreting dynamic situations, students might wonder whether a linear or nonlinear graph could represent a relationship between attributes in a dynamic situation. Carlson et al. (2002) investigated calculus students' covariational reasoning on a five-item assessment that tasked students with sketching graphs of dynamic situations, such as a bottle filling with water. One student in their study sketched an appropriate nonlinear graph, and gave this reason for their sketch: "I just know that it must be smooth because this is what these graphs always look like, not disconnected line segments" (p. 364). This response suggested that the student had an expectation about what the graph *should* look like (Johnson et al., 2020), and this opinion influenced their graph sketching. For example, if students were working on a unit about linear function, they might expect a linear graph to represent relationships between attributes in a dynamic situation. Hence, students' expectations about linear and nonlinear graphs could result from experiences that include only certain kinds of Cartesian graphs.

Graphing conventions (Moore et al., 2014) can influence students' expectations of graph features. For example, if students only encounter graphs that have time as an independent variable on the x-axis, they may think that time is a variable that is always plotted on the x-axis. This can result in students introducing "time" in dynamic situations even when it is only an implicit variable (Kertil et al., 2019; Patterson & McGraw, 2018; Yemen-Karpuzcu et al., 2017). For example, a student aiming to sketch a graph of a relationship between two attributes that vary in their direction of change may instead sketch a graph that has one of these attributes on the y-axis and continuing time on the x-axis. Furthermore, students may exclusively associate dynamic situations involving an object moving at a constant speed with linear graphs and dynamic situations involving an object moving at a non-constant speed with nonlinear graphs (Patterson & McGraw, 2018; Stavley & Vidakovic, 2015). This can be problematic for students' interpretations of dynamic situations that involve linear relationships between attributes of an object moving at a non-constant speed, and vice versa. Hence, students' previous experiences with Cartesian graphs can play a role in their expectations about observable features of graphs.

Methods

This collective case study (Stake, 2005) is part of a larger, National Science Foundation funded project designed to promote mathematical reasoning and instructional transformation in college algebra. This study is a secondary analysis of a larger interview-based validation study (n=31 students) of a fully online assessment, the MGSRDS, designed to measure students' graph reasoning and selection for dynamic situations (Johnson et al., 2021).

The MGSRDS

The MGSRDS assessment consists of six items, each appearing in random order. Each item has four parts, split into two screens. On Screen 1, students are to view a short animation of a dynamic situation involving changing attributes. Then, students are to confirm whether they understood the situation. On Screen 2, students are to select a graph, from among four possible graph choices, to best represent a relationship between attributes in the dynamic situation. Then, students are to explain why they chose the graph that they did.

The six items included a cart on a turning Ferris wheel (Ferris Wheel), a person walking forward and back along a path (Nat & Tree), a fishbowl filling with water (Fishbowl), a toy car moving along a square track (Toy Car), a cone changing in size (Changing Cone), and two insects walking back and forth along a path (Ant & Ladybug). Figure 3 contains still images of the Toy Car item, along with the four graphs choices.

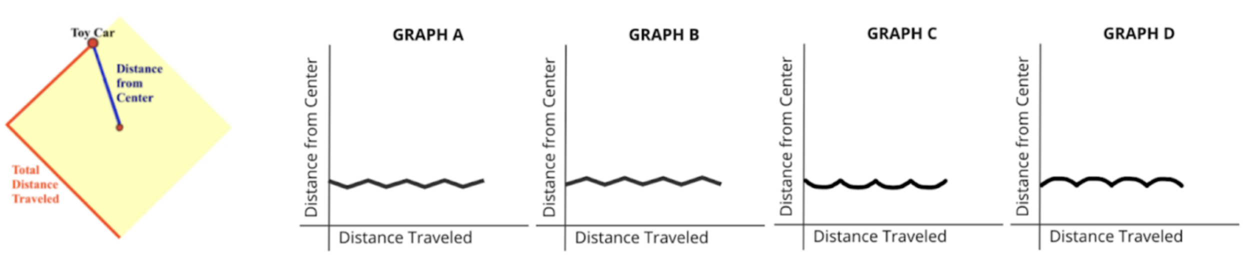

Figure 3: Toy Car Dynamic Situation with Graph Choices

The first five items contain one attribute that varied in its direction of change and one attribute that did not. For example, in the Toy Car item, the toy car's distance traveled always increases, while the toy car's distance from the center increases and decreases. Graph choices for the first five items include two piecewise-linear options and two nonlinear options. Each piecewise-linear graph choice has a nonlinear graph choice that represents the same gross covariation in attributes. For example, Graphs B and D for the Ferris Wheel item (see Figure 2) represent the cart's height increasing, then decreasing, then increasing again, while the distance traveled continues to increase. For the Ferris Wheel item, Graph D is correct because it represents the correct direction of change in attributes and it represents the correct (nonlinear) relationship between attributes (i.e., an amount of change in distance traveled does not equal the same amount of change in height). Graph B is partially correct in that it represents the correct direction of change in attributes, but the incorrect relationship between attributes.

The sixth item, Ant & Ladybug, is different from the other five items in two main ways. First, both attributes in the dynamic situation vary in their direction of change (the Ant's distance from home increases and decreases while the Ladybug's distance from home decreases and increases). Second, the four graph choices do not include two piecewise-linear options and two nonlinear options. There is one graph that is correct, and no graph that is partially correct. For this study, we focus only on the first five items, which included both piecewise-linear options and nonlinear options for the graph choices.

Data Collection and Participants

As part of the MGSRDS validation study, Johnson conducted 31 individual interviews with college algebra students, via Zoom video conference. Students volunteered to participate in the interviews, and they received a gift card for their participation. A graduate research assistant (GRA) attended each interview, to observe and take field notes. After each interview, a GRA created a verbatim transcript.

During the interviews, we noticed that there were students who spontaneously wondered about whether they should select a piecewise-linear or nonlinear graph. Thus, we used purposeful sampling (Gliner et al., 2017) to determine participants. Looking at each of the 31 transcripts, we did a keyword search, using the search terms "linear," "curv," and "straight." From this pool, we identified students who wondered about linearity/nonlinearity across five MGSRDS items (Ferris Wheel, Nat & Tree, Fishbowl, Changing Cone, Toy Car). There were six such students. We selected two students, Emma and Maya (pseudonyms), as cases for this study, because they demonstrated evidence of covariational reasoning across the majority of items, in addition to wondering about whether to select a piecewise-linear or nonlinear graph.

Data Analysis

We started by watching Emma and Maya's video recordings and reading their transcript. Then we analyzed in terms of specific and generic properties, to better understand the phenomenon within and across cases (Stake, 2005). In terms of specific properties, we identified and documented any evidence of four forms of students' graph reasoning (Johnson et al., 2020). In terms of generic properties, we noted any excerpts when students spoke about linear and/or nonlinear graphs. To gain a better overall understanding of each individual case, Knurek created a table describing identified instances of Emma and Maya's graph reasoning and selection, in chronological order, for each of the five MGSRDS items (Ferris Wheel, Nat & Tree, Fishbowl, Changing Cone, Toy Car). Then, Knurek looked across these instances to make inferences about each students' graph reasoning and selection process for each item. After creating these tables, Johnson and Knurek met to vet inferences and come to a consensus.

Results

We report on Emma and Maya's work on two MGSRDS items, Ferris Wheel and Toy Car. Their reasoning on these items is representative of the broader set of items. These items include conventional (Toy Car) and unconventional graphs (Ferris Wheel). Furthermore, these items include graphs having physical features that share resemblances with (Ferris Wheel) and differences from (Toy Car) the dynamic situation. The toy car's track is composed of straight lines, but the correct graph is nonlinear. In contrast, the Ferris wheel is rounded and the correct graph is curved. We organize this section first by each case, outlining how Emma and Maya's graph reasoning relates to their graph selection on each of the two MGSRDS items. Then, we present results from our cross case analysis.

Expecting Graph Features Before Seeing Graph Choices: Emma

The Ferris Wheel Item. On Screen 1, Emma watched the video animation and made a connection to her work in her college algebra class:

> Emma (Screen 1): I believe we had something like this in my algebra class, and it says, we'll focus on the Ferris wheel cart's height from the ground in the total distance traveled. So, we're going to be analyzing a graph and it's not going to be a straight line due to the increasing distance it gets from the ground and then how it gets closer to the ground. But no matter what, it's always going to be increasing in the total distance traveled.

Before even seeing any of the graph choices, Emma spontaneously shared her expectation of the graph being nonlinear, like the graph that she had seen in her algebra class. Her response on Screen 1 shows evidence of gross covariational reasoning (Thompson & Carlson, 2017) because she talked about changes in the Ferris wheel cart's height from the ground and total distance traveled. When she moved on to Screen 2, Emma looked at the four graph choices and immediately eliminated both piecewise-linear graph choices:

> Emma (Screen 2): I would automatically eliminate A and B, because the Ferris wheel doesn't seem to move in a rigidity pattern like that. It is making a circle. So it'd be a curved line. And I'm choosing graph D, because it starts off by going forward, instead of going backwards when you're going, um, vertical. And to me, that just makes more sense because it starts by going up and then it goes down, and then it goes back up.

Emma's justification for eliminating the two piecewise-linear graphs ("the Ferris wheel doesn't seem to move in a rigidity pattern like that") shows evidence of motion reasoning, per the framework from Johnson et al. (2020). When distinguishing between the two remaining graphs, Emma appealed to the direction of change in the height of the Ferris wheel cart, which resulted in her selection of a single graph. In summary, Emma's graph selection process for the Ferris wheel item involved her gross covariational and motion reasoning, along with her prior expectation of the graph being nonlinear.

The Toy Car Item. On Screen 1, Emma demonstrated evidence of gross covariational reasoning, similar to how she did on the Ferris wheel item. Again, Emma spontaneously shared her expectation of the graph, saying, "we're going to look at more of, like, a zigzagged line." Upon moving to Screen 2, Emma eliminated two graphs. However, unlike the Ferris wheel item, the two graphs she eliminated represented the same direction of change in attributes. This left her with two graph choices, one piecewise-linear and one nonlinear (Graphs A and C, see Figure 3). To distinguish between these graphs, Emma said, "I'm going to pick Graph A, because to me, it just makes more sense to move on a straight line when we're talking about a square." This response shows evidence of motion and iconic reasoning, per the framework from Johnson et al. (2020), because she talks about motion of the car and the physical shape of the track. Hence, Emma's graph selection process for the Toy Car item involved her gross covariational, motion, and iconic reasoning, along with her prior expectation of the graph being piecewise-linear.

Determining Graph Features After Seeing Graph Choices: Maya

The Ferris Wheel Item. On Screen 1, Maya watched the video animation, described the situation while showing evidence of gross covariational reasoning, and moved on to Screen 2. Upon looking at the four graphs on Screen 2, Maya eliminated two graphs that she didn't think represented the correct direction of change in attributes:

> Maya (Screen 2): I'm going to eliminate Graphs A and C because if you look at how the height is going it starts off decreasing but, according to the animation, the Ferris wheel started from the midpoint, the height of the cart was at the midpoint and when it started moving, its increased and didn't decrease so, in Graphs A and C show that the cart starts at a decreasing height so those two are out.

Maya's response shows evidence of gross variational reasoning because she talked about changes in the Ferris wheel cart's height. After eliminating these graphs, Maya spontaneously brought up a challenge with distinguishing between "pointed" and "curved" graphs:

> Maya (Screen 2): To be honest, I don't really know the differences between the graphs that are pointed and, and the graphs that are more curved. So, I'm just, I don't know, I'm going to pick Graph D. And this, this isn't because of a really educated guess but since we're dealing with a circular kind of model, I'm assuming that the graph will also look that way.

Maya's response shows evidence of iconic reasoning in her justification for picking a nonlinear graph ("we're dealing with a circular kind of model"). In summary, Maya's graph selection process for the Ferris wheel item involved her covariational, variational, and iconic reasoning.

The Toy Car Item. On Screen 1, Maya again demonstrated gross covariational reasoning after watching the video animation, like she did for the Ferris Wheel item. Upon moving to Screen 2, Maya eliminated two graphs. However, unlike the Ferris wheel item, she eliminated both nonlinear graphs. Her justification showed evidence of motion reasoning: "I feel like the toy car, its movement was just, I don't know, linear-like." When distinguishing between the two remaining graphs, Maya appealed to the direction of change in the toy car's distance from the center of the track, saying "the toy car started at an edge and then moved closer to the midpoints of that first edge, meaning that the distance started off with decreasing." Hence, Maya's graph selection process for the Toy Car item involved her gross covariational and motion reasoning.

Cross Case Analysis: Graph Selection and Reasoning

Physical Graph Features vs. Direction of Change. The order in which Emma and Maya eliminated graphs reversed between the MGSRDS items. On the Ferris Wheel item, Emma eliminated two graphs based on linearity, then selected a single graph based on the direction of change in attributes. In contrast, on the Toy Car item, Emma eliminated two graphs based on the direction of change in attributes, then selected a single graph based on linearity. Maya's graph selection followed a similar process, but the order was switched (i.e., direction of change before linearity on the Ferris Wheel item, and linearity before direction of change on the Toy Car item). Notably, Emma and Maya's conceptions of the direction of change in attributes and their expectations for the graph features informed their graph selection. While this was sufficient to select a correct graph for the Ferris Wheel item, it was insufficient to do so for the Toy Car item.

Physical Features of Situation vs. Physical Features of Graphs. Emma and Maya both selected the correct graph for the Ferris Wheel item, and the partially correct graph for the Toy Car item. They both chose the same graphs on both items, despite differences in the order of their graph reasoning. Notably, the physical features of the Ferris wheel situation (curved wheel) resembled physical features of the correct graph (curved).

Discussion and Conclusion

We investigated how students' graph reasoning related to their graph selection on the MGSRDS items that included piecewise linear and nonlinear graph choices. Our qualitative analysis revealed two approaches through which students' graph reasoning intertwined with their graph selection. The first approach was to narrow down graph choices to two graphs based on physical features (i.e., both piecewise-linear or both nonlinear), then select a single graph via gross covariational reasoning. The second approach was to narrow down graph choices to two graphs based on the direction of change in attributes (i.e., one piecewise-linear and one nonlinear), then select a single graph via motion and/or iconic reasoning. Notably, Emma and Maya used both approaches. Furthermore, when physical features of the dynamic situation resembled physical features of the graph, either approach resulted in the correct graph choice.

Our findings point to affordances and constraints in students' graph selection when students engage in gross covariational reasoning (Thompson & Carlson, 2017). An affordance is to interpret and select graphs based on direction of change in attributes. Both Emma and Maya selected graphs that represented gross covariation in attributes for the Ferris Wheel and Toy Car items. However, to distinguish between two different looking graphs that both represented the same gross covariation in attributes (e.g., Graphs A and C, see Figure 3), gross covariational reasoning alone was insufficient. Because these interviews were part of the MGSRDS validation, Johnson did not engage in exploratory teaching to investigate whether students might extend beyond gross covariational reasoning. Students' questions about graph curvature could prompt their investigation of more nuanced relationships between quantities (Johnson et al., 2017). In future studies, researchers could investigate conditions under which students might shift beyond gross covariational reasoning, to more advanced forms of covariational reasoning.

The framework from Johnson et al. (2020) distinguishes between motion/iconic reasoning and variational/covariational reasoning. This distinction is compatible with Moore and Thompson's (2015) distinction between static and emergent shape thinking. Static shape thinking involves conceiving of graphs as physical objects, while emergent shape thinking involves conceiving of graphs as in-progress traces of relationships between covarying quantities. Our results suggest that students use both physical feature-based (e.g., motion, iconic) and quantitative-based (e.g., variation, covariation) reasoning to select graphs representing relationships between attributes in dynamic situations. In future studies of students' graphing, researchers can examine how students' static and emergent shape thinking may intertwine.

Our findings pointed to the viability of including unconventional graphs (e.g., Moore et al., 2014) on assessment items. When Maya and Emma encountered unconventional graphs (e.g., on the Ferris wheel item), they engaged in gross covariational reasoning to make sense of relationships between attributes. This supported an earlier claim from Moore et al. (2014), that breaking conventions (e.g., On a Cartesian graph, a variable varying consistent with elapsing time is to be represented on the horizontal axis.) could promote students' quantitative reasoning.

In conclusion, it is valuable for students to question whether a graph should be linear or curved. While we are encouraged that Emma and Maya spontaneously raised this question during an interview, we contend that the organization of function topics in many U.S. College

Algebra textbooks (e.g., Sullivan, 2020) can limit such questioning. If students only encounter particular function types in separate units, there is little room for such exploration.

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