An intellectual need for relationships: Engendering students' quantitative and covariational reasoning

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Abstract

Drawing on Harel's construct of "intellectual need" (1998, 2008b, 2013), I propose an expansion in possible categories of such needs, to include an "intellectual need for relationships." This is a need to explain how elements work together, as in a system. Broadly, I offer Freudenthal's (1973) term, "mathematizing," to describe a category of a way of thinking that can emerge from an intellectual need for relationships. I argue that this need can engender students' quantitative and covariational reasoning, important not only for their mathematical development, but also for being informed citizens. I put forward four facets of an intellectual need for relationships, addressing task design considerations for each: attributes in a situation (What are the things?), measurability of attributes (How can things be measured?), variation in attributes (How do things change?), and relationships between attributes (How do things change together?). I conclude with implications for theory and practice.

Keywords: Quantitative Reasoning, Covariational Reasoning, Intellectual Need, Mathematizing, Task Design

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An intellectual need for relationships: Engendering students' quantitative and covariational reasoning

People encounter situations involving change and variation as citizens of the world. For instance, sea levels are rising as the oceans continue to absorb heat from the atmosphere. One may read about this phenomenon in newspaper articles or encounter graphs representing rising sea levels over time. By engaging in quantitative and covariational reasoning (Carlson et al., 2002; Thompson, 1994, 2011; Thompson & Carlson, 2017), people can interpret and make meaning of such situations (e.g., González, 2021). Not only are these forms of mathematical reasoning productive for being informed citizens, but they also underlie key mathematical concepts such as rate and function (Thompson & Carlson, 2017). Hence, it is crucial for students to develop and engage in such reasoning, and for opportunities to occur throughout their schooling, across K-12 and university mathematics courses. Yet, from a student's point of view, what may serve as a catalyst, so students can actualize potential opportunities? Drawing on Harel's construct of "intellectual need" (1998, 2008b, 2013), I offer an intellectual need for relationships, which is a need to explain how elements work together, as in a system. I argue that this need can engender students' quantitative and covariational reasoning.

To illustrate, consider a situation involving Sam, who is walking from home to the corner store. There are a number of attributes that students may separate from the situation; two include Sam's distance from home and Sam's distance from the store. Engaging in quantitative reasoning (Thompson, 1994, 2011), a student can conceive of the possibility of measuring those attributes, even if they do not find particular amounts of measure. For instance, a student may have a sense of a length of a stretchable cord extending from Sam's current location to home or the store. As

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Sam is walking, each distance changes, increasing or decreasing depending on Sam's route. Engaging in covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017), a student can conceive of relationships between the changing distances. For instance, with a direct route, Sam's distance from home increases while the distance from the store decreases. By forming and interpreting relationships between attributes, students can mathematize (Freudenthal, 1973) such situations in terms of quantities and covariation.

Results of researchers' investigations of students' quantitative and covariational reasoning represent both challenge and promise. Even accomplished university students have demonstrated difficulty (e.g., Carlson et al., 2002; Moore, Stevens, et al., 2019), while middle and secondary students have shown promising evidence (e.g., Ellis et al., 2020; Johnson, 2012). I argue that students' intellectual need for such reasoning may account, in part, for differences in these findings. For example, consider a task in which students are to sketch a Cartesian graph relating Sam's distance from home and Sam's distance from the store. Some students may find such a task problematic; they may wonder how to measure and relate the different distances as they sketch their graph. In contrast, other students may think the task is an exercise in finding a resulting graph that is an instance of some familiar graph. If students are focused on getting end results, they may miss opportunities to engage in quantitative and covariational reasoning.

Harel (1998, 2008b, 2013) put forth the construct of intellectual need, rooted in Piaget's constructivist theory. To illustrate, say a student encounters a situation that is problematic for them, and as a result of engaging with that situation, they develop some new mathematical knowledge. The "problematic-ness" of that situation, from the student's point of view, is the student's intellectual need. For example, one student may intend for Sam's graph to represent a relationship between distances. Another student may intend to represent Sam's physical motion

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on the walk. While both students find the situation problematic, the first student's goal is more compatible with quantitative and covariational reasoning.

Harel (2008a) has posited two different forms of mathematical knowledge that can emerge from students' intellectual needs: ways of understanding (products of mental action) and ways of thinking (characteristics of mental action). For example, a conception of function can be a product of mental action, and a correspondence approach can be a characteristic of mental action. Through broad categories, Harel has illuminated three ways of thinking (2008a) and five forms of intellectual need (2013), leaving room for the possibility that more categories can emerge. I argue for an expansion of the ways of thinking and forms of intellectual need put forward by Harel.

I organize this chapter into six sections. First, I discuss theoretical underpinnings of quantitative and covariational reasoning. Second, I offer Freudenthal's term, "mathematizing" (Freudenthal, 1973), to represent an additional category of a way of thinking that can emerge from students' intellectual need. Third, I explain what I mean by an intellectual need for relationships, and how that need may engender students' quantitative and covariational reasoning. Fourth, I put forward four facets of such a need. Fifth, I address task design considerations for each facet, using a digital Ferris wheel task to illustrate. Sixth, I discuss implications for theory and practice.

Theorizing Quantitative and Covariational Reasoning

Thompson rooted the theory of quantitative reasoning (1994, 2011) in Piaget's constructivist theory, which assumes that individuals develop new understandings by reorganizing their existing conceptions. From this lens, the distances I identified in the situation of Sam walking from home to the store would not be "out there" for a student to observe. Rather,

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they would be a person's conception of the situation. In the theory of quantitative reasoning, Thompson explains how individuals may conceive of situations in terms of attributes that are possible to measure, such as the distances in Sam's situation. Engaging in quantitative reasoning involves conceptions of quantities, a quantification process, and quantitative operations. A student's quantitative reasoning can entail some or all of these elements.

Quantities are a foundational element of the theory. Per Thompson (1994), a quantity is an individual's conception of an attribute in a situation as being possible to measure. This means that quantities are human creations; through their conceptions, individuals transform attributes into quantities. For example, in Sam's situation, a student can transform attributes into quantities by separating those attributes (e.g., distance) from the physical motion described in the situation (e.g., Sam's walking). Essential to Thompson's theory is a distinction between conceiving of the possibility of measurement and the act of determining particular amounts of measure. This means that students can think of measuring Sam's distance from the store without finding certain amounts of distance.

With quantification, Thompson (2011) explained a three-part process by which an individual can formalize this "possible to measure-ness." First, they would conceive of an attribute that could be measured, such as Sam's distance. Second, they would conceive of a unit of measure for the attribute. This might be a standard unit, such as a meter or foot, or a nonstandard unit, such as one of Sam's steps. Third, they would conceive of a proportional relationship between the unit and the attribute's measure. That is, they could iterate one of the units, such as a step length, to measure Sam's distance from the store. As with quantity, an

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essential aspect of quantification was that an individual did not need to actually measure Sam's distance from the store with the indicated unit, just think of the possibility of doing so.

Thompson (1994, 2011) put forward quantitative operations to describe mental activity in which individuals could employ a quantitative lens on situations and conceive of new kinds of quantities. Thompson identified a "difference" as one such quantity that students could create via additive comparison. For example, at any instant in Sam's walk, a student might compare Sam's distance to the store and Sam's distance from home to create a new quantity, the difference between the distances. As with quantity and quantification, students could engage in quantitative operations without determining particular amounts of difference.

With Figure 1, I express interconnections between quantity, quantification, and quantitative operations. Because both quantification and quantitative operations extend from quantities, I have placed unidirectional arrows between quantity and those elements. Conceiving of an attribute as being possible to measure is the first part of the quantification process. By engaging in quantitative operations, students can create new quantities in relationship to quantities they already know. I have placed bidirectional arrows between quantification and quantitative operations to indicate a reflexive relationship between them. Students can engage in quantification of some quantities, create new quantities, and then engage in quantification yet again.

[Figure 1 goes here]

Covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017) entails conceptualizing relationships between attributes, which individuals perceive to be capable of varying and possible to measure. For example, one student may conceive that Sam's distances change in harmony with each other, their values continually changing together: "Sam's distance

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from home increases while Sam's distance from the store decreases." In contrast, another student may think about snapshots of the distances at particular instances in Sam's walk: "Now Sam is 2 blocks from home and 18 blocks from the store; now Sam is 5 blocks from home and 15 blocks from the store." As suggested by these examples, when individuals conceive of variation, or covariation in attributes, they have mental images of how those attributes have changed or are changing. Castillo-Garsow (2012) proposed the terms "chunky" and "smooth" to distinguish these images of change. One way to conceive of a distinction between these images is in their "countable-ness." Chunky images entail countable units, whereas smooth images entail a continual flow of change (Castillo-Garsow et al., 2013). The first example suggests smooth thinking because Sam's distances are continually changing together. The second example suggests chunky thinking because the focus is on particular instances in Sam's walk. While both images of change have utility, there is something special about smooth images of change, which comprise conceptions of continual change in attributes.

Not only did Thompson and Carlson (2017) position smooth thinking at the highest levels of variational and covariational reasoning, they contend that opportunities for students to engage in such thinking should happen early and often. Given the centrality of quantitative and covariational reasoning for students' mathematical development (Thompson & Carlson, 2017), I argue that they are more than just a means to promote students' learning of new mathematical concepts, such as rate or function. They are worthy ways of reasoning in and of themselves.

Mathematizing as a Way of Thinking Emerging from Students' Intellectual Need

To sketch out a landscape of ways of thinking, Harel (2008a) has provided three different, yet interrelated categories: proof schemes, problem-solving approaches, and beliefs about mathematics. Broadly, these ways of thinking involve how people determine the viability of an assertion, think while solving problems, and view mathematics itself. I posit that students' quantitative and covariational reasoning point to a categorically different way of thinking from those put forward by Harel. By engaging in quantitative and covariational reasoning, students can conceive of the possibility of measuring different attributes in a situation and form and interpret relationships between those attributes. This may or may not involve assessing the truth of an assertion, solving a given problem, or considering the nature of mathematics itself. I offer Freudenthal's (1973) term, "mathematizing," to characterize this fourth category of a way of thinking.

When mathematizing a situation, people conceive of some "thing" from a mathematical lens (Freudenthal, 1973). For example, I have provided descriptions of different ways students might mathematize Sam's situation, from quantitative and covariational lenses. These lenses are not "out there" for people to see, rather they are ways of thinking that people bring to a situation. By positioning mathematizing as a complementary, yet distinct way of thinking, from those put forward by Harel (2008a), I foreground mental actions involved in this human activity.

To provide a rationale for this fourth category, I appeal to the construct of goals (Simon & Tzur, 2004). By goal, I mean some achievable outcome that a person has set in an educational setting, rooted in their current conceptions and tasks they encounter (Simon & Tzur, 2004). A person's goal is a goal from their perspective; it can be different from a teacher or researcher's goal. For example, a teacher may intend for a student to sketch a graph of Sam's situation. One student may have a goal of sketching a graph that shows Sam's movement from home to the store. Another student may have a goal of exploring how Sam's distances are changing together. While the first student has a goal of solving the problem, the second student's goal involves investigating relationships between attributes in the situation. Mental actions compatible with the

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second student's goal are crucial for mathematizing Sam's situation in terms of quantity and covariation.

Like a student's goal, with the construct of intellectual need, a researcher employs their perspective of a student perspective, because an intellectual need is from the perspective of the person engaging in the thinking, rather than an outside observer. I conceive of a person's intellectual need as akin to a goal, with the caveat that an intellectual need emerges when a person finds a situation to be problematic for them. Whereas, a student may have a goal without problematizing anything. For example, one student may sketch a graph with the goal of showing Sam's literal movement, get feedback that a correct graph looks different, and experience nothing problematic about the situation. In contrast, another student with the same goal may wonder what could account for a graph's different features, and adjust their goal based on their wondering. The adjustments may entail separating the attributes from the situation and conceiving of how those attributes might be measured. As suggested by these examples, students may have the same "task" presented to them, yet they can conceive of that task in very different ways.

Students' covariational reasoning has potential to serve as a catalyst for their intellectual need. In a study of two university students, Paoletti and Moore (2017) have shown how aspects of students' covariational reasoning can engender an intellectual need for conceiving of a quantity, such as time, which may only be implicit in a problem situation. In particular, they have found that students conceived of time in a conceptual way (Thompson & Carlson, 2017), not just as elapsing, but as something possible to measure, in terms of duration. As Paoletti and Moore (2017) argued, such a conception can promote students' understanding of parametric function and represents something beyond mental actions in covariational reasoning. Describing this

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finding in terms of students' intellectual need can go like this: A way of reasoning (covariational reasoning) can create an intellectual need for a way of understanding (a new understanding of a quantity implicit in a situation), which can promote students' development of mathematical concepts (parametric function).

In light of the foundational nature of quantitative and covariational reasoning, I posit they are not only catalysts for, but also products of students' intellectual need. With Freudenthal's "mathematizing," I have described such ways of reasoning in broad terms, to illuminate a new category beyond the three offered by Harel (2008a). Broadening categories of ways of thinking can, in turn, make room for new categories of intellectual need. To this end, I propose an "intellectual need for relationships," which can engender students' quantitative and covariational reasoning.

An Intellectual Need for Relationships

Leaving room for the possibility of expansion, Harel (2013) put forward five categories of intellectual need: certainty, causality, computation, communication, and structure. Harel (2013) defined certainty as a need to determine the truth of some conjecture, causality as a need to explain why some assertion is true, computation as a need to determine values of measurable attributes in a situation, communication as a need to formalize and formulate mathematical ideas, and structure as a need to reorganize what is known in a logical way. Together, these intellectual needs provided a landscape to explain how students may reconcile situations they find to be problematic for them.

Students' quantitative and covariational reasoning point to a new category of intellectual need, beyond those put forward by Harel. Broadly, this new category involves a desire to explain, which shares some similarities with a need for causality. A key difference lies in the

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object of the explanation. This new category reflects a need to explain a situation from a mathematical lens, which does not necessitate explaining why something is true or determining the truth of a proposition. Furthermore, this new category of need is different from computation. As Thompson (1994, 2002, 2011) has asserted, students can conceive of the possibility of measuring attributes without finding particular amounts of measure. Forming and interpreting relationships between attributes does not necessitate formalizing ideas into symbolic expressions, or formulating those expressions into the spoken word. Although a need to reorganize existing structures may follow from students' quantitative or covariational reasoning, it addresses a different kind of problem. Thus, I offer a sixth category of intellectual need: relationships.

Harel (2013) has posited that intellectual needs have three main characteristics, which I summarize here. First, they are from the perspective of a person, not an outside observer such as a researcher or teacher. Second, they are something people learn, not something innate. Third, they are linked to a person's desire to resolve some "problematic-to-them" situation. I view this third characteristic to be a key distinction between goals and intellectual needs. Goals may or may not result from a desire to resolve a problematic situation; they may just be part of engaging in some task. Intellectual needs resolve something problematic for a learner. Laying out each category of intellectual need, Harel (2013) has provided a three-part discussion: definition of the need, description of inchoate conceptions underlying the need, and historical evidence to account for the need. I follow this approach.

An intellectual need for relationships is a need to explain how elements work together, as in a system. This may apply to scientific phenomena, such as global warming, or to everyday situations, such as a filling bottle. While a need for causality is a need for directionality (e.g., A leads to B), a need for relationships is a need to understand how A and B work together. For

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instance, in the classic filling bottle problem (Shell Centre for Mathematical Education (University of Nottingham), 1985), students are to sketch graphs representing the height and volume of liquid in differently shaped bottles. I view an intellectual need for relationships to stem from people's desire to form connections between attributes, so they may mathematize the world around them. Across history, humans have connected measures of attributes, such as the length of an object's shadow, with a duration of time (Barnett, 1999). In the history of mathematics, a need for relationships has played a role in mathematicians' conceptualization of what is now called function.

Appealing to historical accounts of Boyer, which were compatible with those of Kleiner, Thompson and Carlson (2017) identified four broad eras in the evolution of the idea of function: proportion, equation, function (I), and function (II). In their discussion, Thompson and Carlson (2017) threaded the representation of relationships throughout the eras. In the proportion era, "people represented relationships between quantities geometrically" (p. 421). The equation era was "characterized by the use of equations to represent constrained variation in related quantities' values" (p. 422). The first function era was "characterized by explicit representations of a relationship between values of two quantities so that values of one determined values of another" (p. 422). The second function era, which is still continuing, was "characterized by values of one variable being determined uniquely by values of another" (p. 422). Thompson and Carlson (2017) emphasized how ideas of variation and covariation were central to people's development of the function concept, even though the evolution of people's meaning for function relegated those ideas to lesser, or even seemingly absent, roles.

In reflecting on the discussion of Thompson and Carlson (2017), I note a shift in the foreground and background, coinciding with the invention of algebraic representations. As

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algebraic representations have become more formal, causality has come to the foreground (e.g., the possibility of determining one variable's value given another). In turn, an explanation of how elements in a system work together has faded to the background (e.g., relationships between quantities given constraints in their variation). By proposing a need for relationships, I mean to foreground ways of reasoning, including quantitative and covariational reasoning, which are crucial for students' mathematical development (Thompson & Carlson, 2017).

Four Facets of an Intellectual Need for Relationships

I put forward four facets of an intellectual need for relationships: attributes in a situation (What are the things?), measurability of attributes (How can things be measured?), variation in attributes (How do things change?), and relationships between attributes (How do things change together?). I think of an intellectual need like a mental gemstone; a sparkling, multifaceted conception that can illuminate things once puzzling or mysterious. I include a question with each facet to emphasize a person's point of view, what they may be wondering in a situation. The first two facets address quantitative reasoning and the mental action of quantification. The last two address variational and covariational reasoning, respectively. I view the first facet, attributes in a situation, to ground the other facets, because it focuses on the "things" which people can separate from a situation, then conceive of as possible to measure or capable of varying.

While I present four facets, I leave room for the possibility for more to be included. I propose these facets based on theoretical underpinnings of quantitative and covariational reasoning, and on empirical results of fine-grained studies that I have led to investigate middle and secondary students' reasoning. My colleagues and I have found these facets to illuminate students' progressions in (or challenges with) their engagement in covariational reasoning (Johnson et al., 2020; Johnson, McClintock, et al., 2017; Johnson & McClintock, 2018). I

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describe conceptions related to each facet, then highlight results to demonstrate how those conceptions were (or could have been) productive for students' reasoning.

Attributes in a Situation: What are the Things?

To conceive of relationships between elements in a system, people need to separate those elements, or attributes, from the system itself. A conception of attributes themselves is foundational to quantitative (and covariational) reasoning. For example, to begin quantifying Sam's situation, students would separate attributes, such as Sam's distances from home and the store, from the physical situation itself. This conception may sound too obvious to highlight (e.g., of course students will separate distances from a situation); however, students' long-standing challenges with sketching and interpreting graphs suggest otherwise. Two enduring challenges involve conceptions of graphs as providing a static picture of a situation (Leinhardt et al., 1990), such as a physical map, or as showing the physical motion in a situation (Kerslake, 1977).

If students approach a graphing task with a goal of representing the physical motion they perceive in a situation, they likely will sketch a graph inconsistent with constraints of a Cartesian coordinate system. For example, a student may expect that a graph of Sam's walk from home to the store should share some physical characteristics with Sam's journey, and in turn, that student may sketch a graph that resembles the literal path that Sam took, regardless of the distances labeled on the axes. Even in the face of such inconsistencies, this goal can remain persistent for secondary students (Johnson et al., 2020).

Measurability of Attributes: How can Things be Measured?

To explain how elements work together in a system, people can mathematize different elements, or attributes, in a situation. I focus on a person's conception of the possibility of measuring some attribute they have separated from a situation, or their conception of a quantity, per Thompson's theory. Such a conception may or may not entail all three aspects of Thompson's (2011) process of quantification. For instance, a student may think of Sam's distance from home as a path represented by a line drawn on a map or a trail of breadcrumbs Sam may have left while walking. This student is doing more than just thinking of Sam engaging in the physical activity of walking to the store. They are separating an attribute from the situation and conceiving of how they might measure it. Students can extend from this conception to engage in all aspects of the quantification process by conceiving of a unit of measure and a multiplicative relationship between the unit of measure and the attribute.

When students conceive of how attributes may be measured, they are in a ripe position to mathematize variation in attributes. Evan McClintock and I have found that when middle school students conceived of an attribute as being possible to measure, it impacted their conceptions of variation in that attribute (Johnson & McClintock, 2018) when interacting with dynamic computer tasks involving "filling" polygons. For example, one task was a "filling triangle," in which students were to watch an animation of a right triangle "fill" with color, moving vertically from its horizontal base to the opposite vertex. All students who discerned variation in unidirectional change in that attribute (e.g., The "fill" increases, but the increases are slowing.) were those who conceived of the triangle's "fill" as an attribute possible to measure (e.g., the area of a polygon).

Variation in Attributes: How do Things Change?

When exploring how elements work together in a system, students can conceive of how those elements, or attributes, vary. I liken this to a conception of a variable as something whose values can vary, rather than just a placeholder for some unknown value. When students engage

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in smooth thinking, they can conceive of continual variation in an attribute. Yet, at some point, there is reason to stop a continuation of ongoing values, at which point a person can conceive of some accumulated amount (Castillo-Garsow et al., 2013). Johnson (2012) used the term "smooth chunks" to describe products of this way of thinking, to distinguish them from "chunks" emerging from chunky images of change. It is productive for students to have space to wonder, "How do things change?," before determining, "By how much have things changed?," because they can conceive of values in an interval, and not just find beginning and ending amounts.

When students conceive of continuing variation in individual attributes, it is a productive time for teacher/researchers to engage them in tasks to promote their covariational reasoning. In two different studies that I have led, with secondary students from different school settings, when students conceived of continuing variation in individual attributes in a situation, they were able to shift to covariational reasoning via their work on digital task sequences (Johnson et al., 2020; Johnson, McClintock, et al., 2017). Not only did students shift their reasoning, but they also were aware of a change in their thinking and found the new way of thinking to be powerful for them.

Relationships between Attributes: How do Things Change Together?

To explain how elements work together in a system, students conceive of how those elements change together, forming and interpreting relationships between those attributes. Put another way, they engage in covariational reasoning. Such reasoning can promote students' conception of nuances in relationships between attributes. For example, a student may wonder why a graph has a particular kind of curvature, or whether a linear or nonlinear graph may best represent a relationship. This can allow students to fine-tune their interpretations of graphs, and they can discern "new-to-them" distinctions.

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Secondary students' engagement in covariational reasoning can foster their attention to and accounting for distinctions and nuances in graphs that represent relationships between attributes in linked animations (Johnson et al., 2020; Johnson, McClintock, et al., 2017). I discuss two instances, in which students had a spontaneous question or noticing, during an individual task-based interview. These instances illustrate how a student's intellectual need for relationships can intertwine with their graphing.

One student, Alan, spontaneously questioned how it could be possible for a graph to be piecewise linear, when he noticed the linked animation was slowing down (Johnson et al., 2020). The graph represented a relationship between distance and height, with time as an implicit variable, because each of the attributes were varying with elapsing time. The researcher invited Alan to explore the situation further. Relating different amounts of distance and height, Alan convinced himself that a piecewise linear graph best represented the relationship (Johnson et al., 2020). Another student, Ana, spontaneously noticed differences in the curvature between a graph that she had drawn and a computer graph. (Ana's graph looked more like a parabolic arch and the computer graph was a sine curve.) By conceiving of how two different attributes were varying together in the situation (e.g., The Ferris wheel cart is gaining a lot of distance, but only a little bit of height), she decided it made more sense for her graph to curve in a way that would account for that kind of covariation (Johnson, McClintock, et al., 2017).

Task Design Considerations: A Ferris Wheel "Techtivity"

By a task, I mean something more than an artifact, such as a problem written on paper or a computer activity. Tasks include the intentions and activities of those designing the task, implementing the task, and engaging with the task (Johnson, Coles, et al., 2017). To illustrate task design considerations, I provide an example of a task, a dynamic computer activity, that my

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colleague, Gary Olson, termed a "Techtivity." My purpose is to illuminate how task designers may work to nurture students' intellectual need for relationships, and in turn engender students' quantitative and covariational reasoning.

The Techtivity that I share is part of a set of seven freely available digital tasks (Desmos, n.d.), usable by a broad range of instructors. Each Techtivity consists of a series of screens which students can move through at their own pace. There are four main components. First, students are to view an animation of a dynamic situation involving change in progress, a move common for researchers designing tasks to investigate students' conceptions of change and variation. Second, students are to manipulate dynamic segments representing measures of two attributes in the situation, a move that operationalizes Thompson's (2002) recommendation that students use their fingers as tools to represent change in individual attributes. Third, students are to sketch a Cartesian graph to represent a relationship between variables in the situation. Fourth, students are to repeat the second and third components for the same situation, with attributes represented on different axes, a move that shares similarities with tasks designed by Moore and colleagues (e.g., Moore et al., 2014). In addition, at the end of each Techtivity there are questions designed to promote students' reflection on relationships between attributes in the situation.

Figure 2 depicts the four components of a Ferris wheel Techtivity. The animation shows a green cart that begins in the middle left of the Ferris wheel (Figure 2, bottom left), and moves clockwise around the Ferris wheel for one rotation. Throughout my description of this Techtivity, I use parentheticals to highlight how the design components link back to the four facets of an intellectual need for relationships.

There are many attributes which students may conceive of in the Ferris wheel situation (What are the things?). For this Techtivity, students are to consider two attributes: the cart's

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width from the center and the cart's distance traveled around the wheel. The width is measured by the cart's horizontal distance from a vertical line extending through the center of the wheel (How can things be measured?). While the width might seem like an arbitrary attribute, people riding on a Ferris wheel may feel this "width" as a sensation of moving out and back while the cart goes around the wheel. Figure 2 (top left) shows a trace of each attribute for a partial turn of the wheel. When students represent change in the width and distance (How does each thing change?), at first the attributes are represented on the vertical and horizontal axes, respectively (Figure 2, top right). Both Cartesian graphs are shown at the bottom of Figure 2. In the second graph (bottom right), the width and distance are represented on the horizontal and vertical axes, respectively (How do things change together?).

[Figure 2 goes here]

At the end of this Ferris wheel Techtivity, there are two different types of reflection questions. One is an interpretation of a single point on the graph, and the other is a comparison of two different graphs. First, students are to predict the green cart's location on the wheel given a point on the graph (Figure 3). Second, they are to determine whether they agree or disagree with a student's claim that the two different looking graphs generated by the computer can represent the same relationship between attributes (Figure 4). In addition, Figures 3 and 4 include selected responses from undergraduate students enrolled in a College Algebra course.

[Figure 3 goes here]

[Figure 4 goes here]

The selected responses to each reflection question provide examples of how students may conceive of the four different facets in their work on the task. Responses to the first reflection question (Figure 3) represent a range of reasons given by students who predicted the cart to be at

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the bottom of the wheel, and mentioned both width and distance in their response (What are the things? How can the things be measured?). Some students have provided specific amounts of width and distance to support their responses (e.g., "0 width from the center," "around ¾ distance travelled"), while others have discussed in more general terms (e.g., "no width," "a lot of distance"). The two responses to the second reflection question (Figure 4) give evidence of students' conceptualization of change in, and relationships between attributes, even when they provide differing views of Val's claim (How is each thing changing? How do things change together?).

With the reflection questions, I have intended to invite students to relate, or even mentally fuse different attributes to make sense of the situation (How do things change together?). Put another way, I have worked to create space to engender students' mental forming of multiplicative objects (Thompson & Carlson, 2017). In the first question (Figure 3), the point students are to interpret is on the vertical axis, representing a location when the green cart's width from center is equal to zero. There are two possibilities, the top and bottom of the wheel. By taking into account both the width from the center and the distance traveled, students can determine the point to represent when the cart is located at the bottom of the wheel. In the second question (Figure 4), the two graphs look different, with the second graph being unconventional, yet they represent the same relationship between attributes. Even advanced university students can have challenges interpreting function relationships represented by unconventional graphs (Moore et al., 2014). With this in mind, I have designed the reflection question in terms of whether students agree or disagree with Val, rather than whether Val is right or wrong. With this move, I intend to make room for students to consider Val's statement as a sensible claim made by a human, rather than rushing to a judgment of the validity of Val's claim. For instance,

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students may think Val's claim is reasonable, yet state Val is wrong, because they do not think such a claim is viable to make in a mathematics class, given the unconventional looking graph.

Discussion

I have posited an expansion in Harel's categories of students' intellectual need, to include a need for relationships; a need to explain how elements work together, as in a system. While interconnected, this need is distinct from the needs that Harel (2013) offered (certainty, causality, computation, communication, and structure). Thompson and Carlson (2017) have discussed how relationships are woven throughout scholars' development of the function concept. Yet, the relationships are something more than just a stepping stone in students' development of the concept of function. Variational and covariational reasoning are theoretical constructs, ways of thinking that can explain students' conceptualizations of situations in ways that are both quantitative and dynamic (Thompson & Carlson, 2017).

When engaging in quantitative reasoning, students mathematize attributes, by conceiving of them as being possible to measure. Both Thompson and Harel discuss mental actions of quantifying and quantification, drawing on Piaget's theory. Harel (2013) explains quantifying in broad terms, such that a person could transform some perceptible "thing," for example a feeling of movement, into a measurable attribute. Thompson's (2011) definition of quantification illuminates three mental actions in such a transformation: a conception of an attribute as possible to measure, a unit with which to measure the attribute, and a multiplicative relationship between the unit and attribute. Harel (2013) has located quantifying within an intellectual need for computation, yet quantifying is not limited to a need for computation. Quantifying entails relationships, which Thompson's definition addresses. By positioning an intellectual need for

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relationships as a need in and of itself, I aim to raise the status of quantifying, and related forms of reasoning, to position them as something more than a means to compute a result.

When engaging in covariational reasoning, students form and interpret relationships between attributes they can conceive to be capable of varying and possible to measure. Images of change are part of such mental action, and those images make a difference. Chunky images of change involve only beginning and ending amounts, while smooth images of change allow for all values in an interval. Accordingly, Thompson and Carlson (2017) position smooth images of change at the highest levels of variational and covariational reasoning. Furthermore, they assert that students' opportunities to engage in smooth thinking come early and often. With the Ferris wheel Techtivity, I provide an example of a task designed to engender such opportunities.

Nurturing students' intellectual need for relationships may help them to develop further abstractions. One possibility is an "abstracted quantitative structure" (Moore et al., this volume; Moore, Liang et al., 2019). Moore, Liang and colleagues (2019, p. 1879) have characterized such a structure as "a system of quantitative relationships a person has interiorized to the extent they can operate as if it is independent of specific figurative material (i.e., representation free)." To illustrate, they report on a preservice mathematics teacher who conceived of the inverse sine function as a relationship that was irrespective of a particular representation type. Questions, such as the second reflection question in the Ferris wheel Techtivity (Figure 4), can afford opportunities for students to conceive of relationships that remain invariant, even if physical characteristics of graphs vary. Both students' responses (Figure 4, right), provide evidence of their conception of relationships between attributes in the situation. Physical artifacts, such as Cartesian graphs, are products of a representation system. I conjecture that students who conceive of relationships between quantities in ways that are independent of such artifacts, can

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discern aspects of the representation system itself (see also Johnson, 2020). Integrating multiple theoretical lenses can be productive for researchers investigating students' quantitative and covariational reasoning while engaging in tasks involving socially shared artifacts, such as Cartesian graphs.

An intellectual need for relationships can serve as a starting point to reimagine curricular materials focused on function. One consequence can be opportunities to conceive of quantities as covarying, for which Thompson and Carlson (2017) advocate. A second consequence can be the ways in which students encounter different types of functions. In U.S. secondary math classrooms, students typically see, in a particular order, different types of functions and graphs representing those function types (e.g., linear, then quadratic, then exponential). With such an approach, students may miss out on the relationships themselves. Our field has spent much time arguing about the order in which to present different function types (e.g., Should linear functions come first? Should exponentials come before quadratics?). I recommend reframing the argument. Rather than organizing materials around function types, center relationships between attributes, then introduce different function types as a way to explain different kinds of relationships.

Conclusion

I proposed an expansion in Harel's categories of intellectual needs and ways of thinking. With my choice of Freudenthal's term, mathematizing, I intended to communicate that conceiving of some "thing" from a mathematical lens was a viable way of thinking in its own right. "Mathematizing" is more than a part of a problem-solving approach or a proof scheme. It is a way of thinking that can entail conceptions of variables as actually varying together, a fundamental mathematical idea not to be backgrounded in service of a formal definition. For students to develop quantitative and covariational reasoning across K-12 mathematics, for which

Thompson and Carlson (2017) advocate, it is important that ways of thinking be positioned as just as valuable as ways of understanding. While worthwhile, such a goal is at odds with high-stakes testing pressure rampant in K-12 education in the U.S. in which students and teachers can hear messages that test results, and consequently answer finding and computation, are the only things that matter. Hence, stakeholders have work to do at multiple levels so that each and every student can have opportunities to engage in mathematical reasoning in spaces where they feel safe and valued.

There is a "tension of intention" (Johnson, Coles, et al., 2017) with task design to engender students' quantitative and covariational reasoning, taking into account students' intellectual need for relationships. While some students may have goals consistent with satisfying an intellectual need for relationships, other students may have different goals. As task designers, it is important to wrestle with the tension of anticipated versus actual intellectual needs in students' engagement with a task situation. As researchers, it is crucial to critique one's own task and research design, to guard against deficit approaches in investigations of student cognition (Johnson et al., 2020).

Promoting students' reasoning and sense making in mathematics classrooms is not a neutral activity. Despite the utility of an intellectual need for relationships, students may not perceive mathematics classrooms to be places where they could exert such a need. They may have internalized that to "play the game" (Gutiérrez, 2009) of mathematics, answers are what matter. Furthermore, even if instructors make space for students' reasoning, existing classroom power dynamics can become more apparent, for example, which student voices get amplified (or marginalized). This can be compounded if the intended reasoning is something that their instructors still need to develop. Thompson and Carlson (2017) have called for research

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investigating teachers' experiences fostering students' covariational reasoning, highlighting how teachers may need to develop such reasoning themselves. Such a problem is complex, and I argue that its investigation could benefit from teams of researchers coordinating different theoretical lenses to explain multiple phenomena at play.

Broadly, I view an intellectual need for relationships to be compatible with broader needs for play and exploration in mathematics. When students are mathematizing a situation via their quantitative and covariational reasoning, they conceptualize that situation in terms of attributes which they conceive to be measurable and the ways in which those attributes can change together. They can play with different possibilities and explore results of their efforts (e.g., What happens if?). Students' quantitative and covariational reasoning are more than just a means to develop their function understanding. When their intellectual need for relationships is nurtured in mathematical spaces, students can feel that their ways of reasoning, as well as the results of their reasoning, are welcomed and valued.

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Figure 1

Key Elements of Thompson's Theory of Quantitative Reasoning

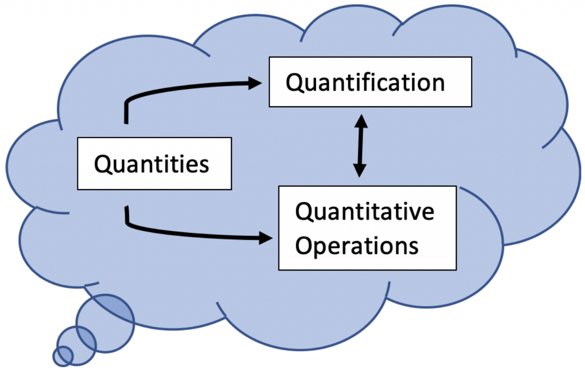

Figure 2

Components of a Ferris Wheel Techtivity

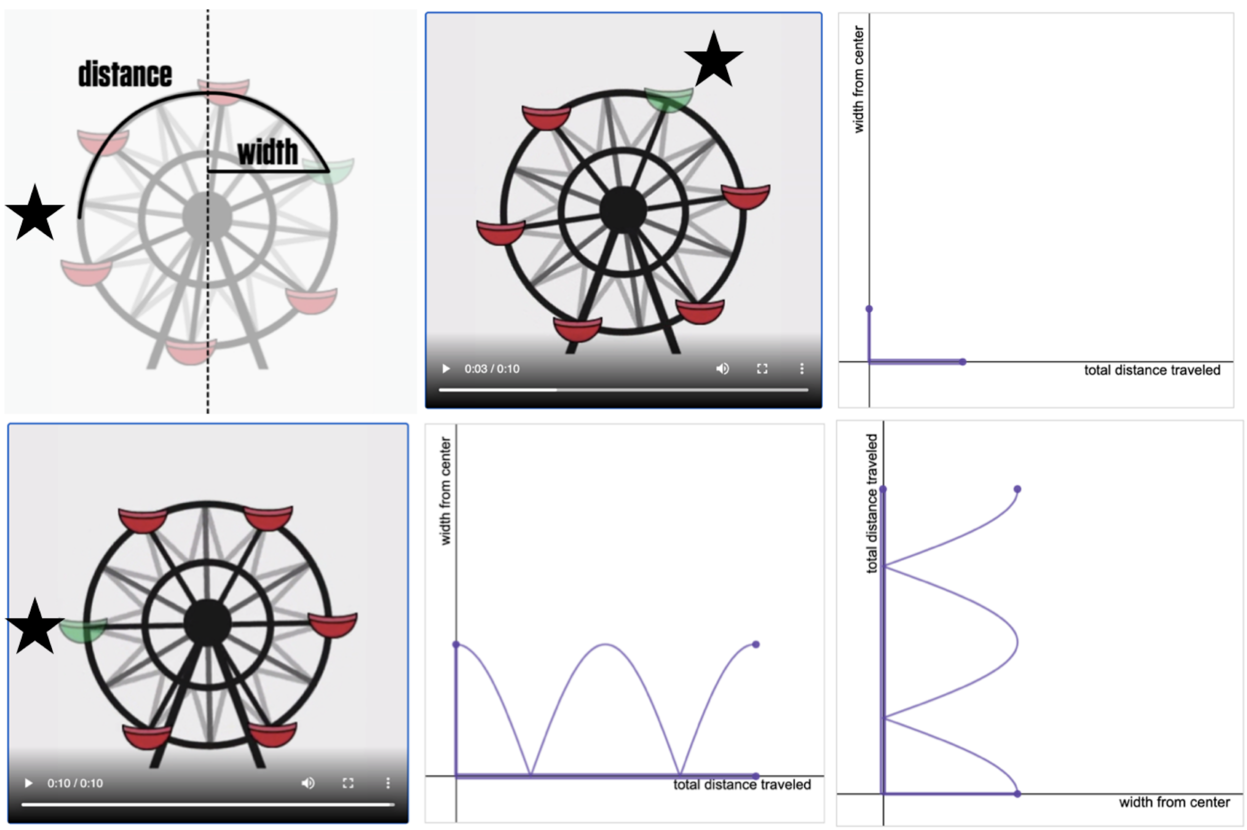

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Figure 3

Reflection Question: Graph Shows a Single Point

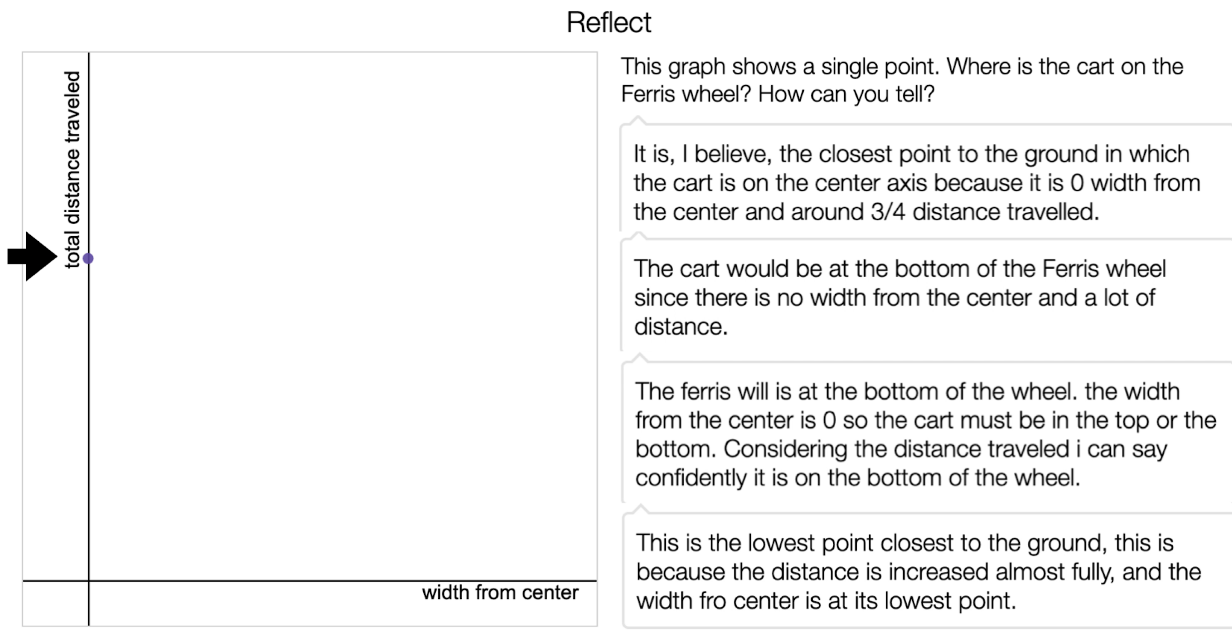
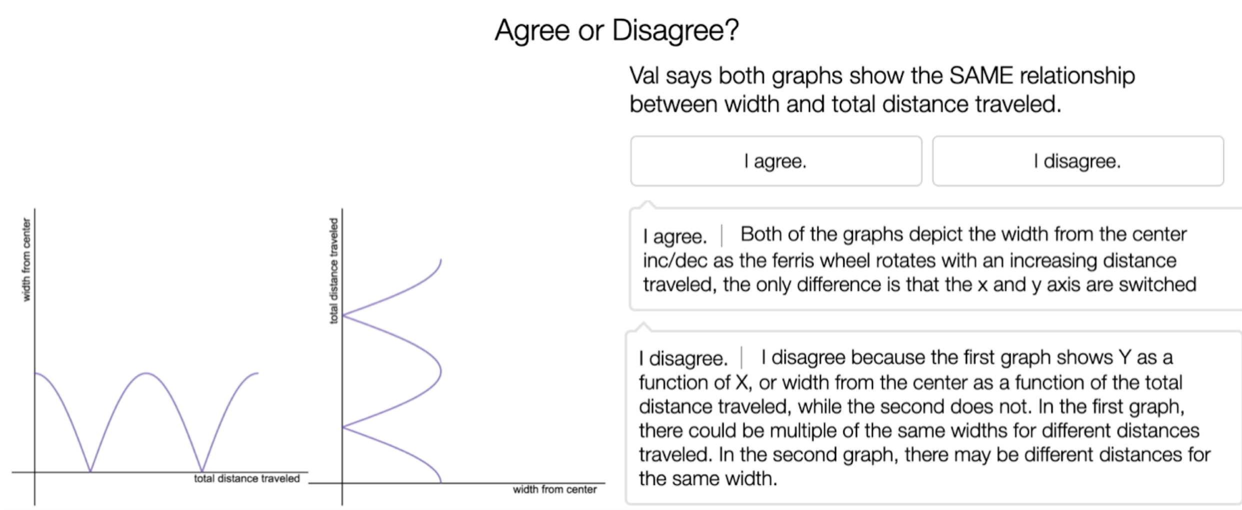


Figure 4

Reflection Question: Agree or Disagree?



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